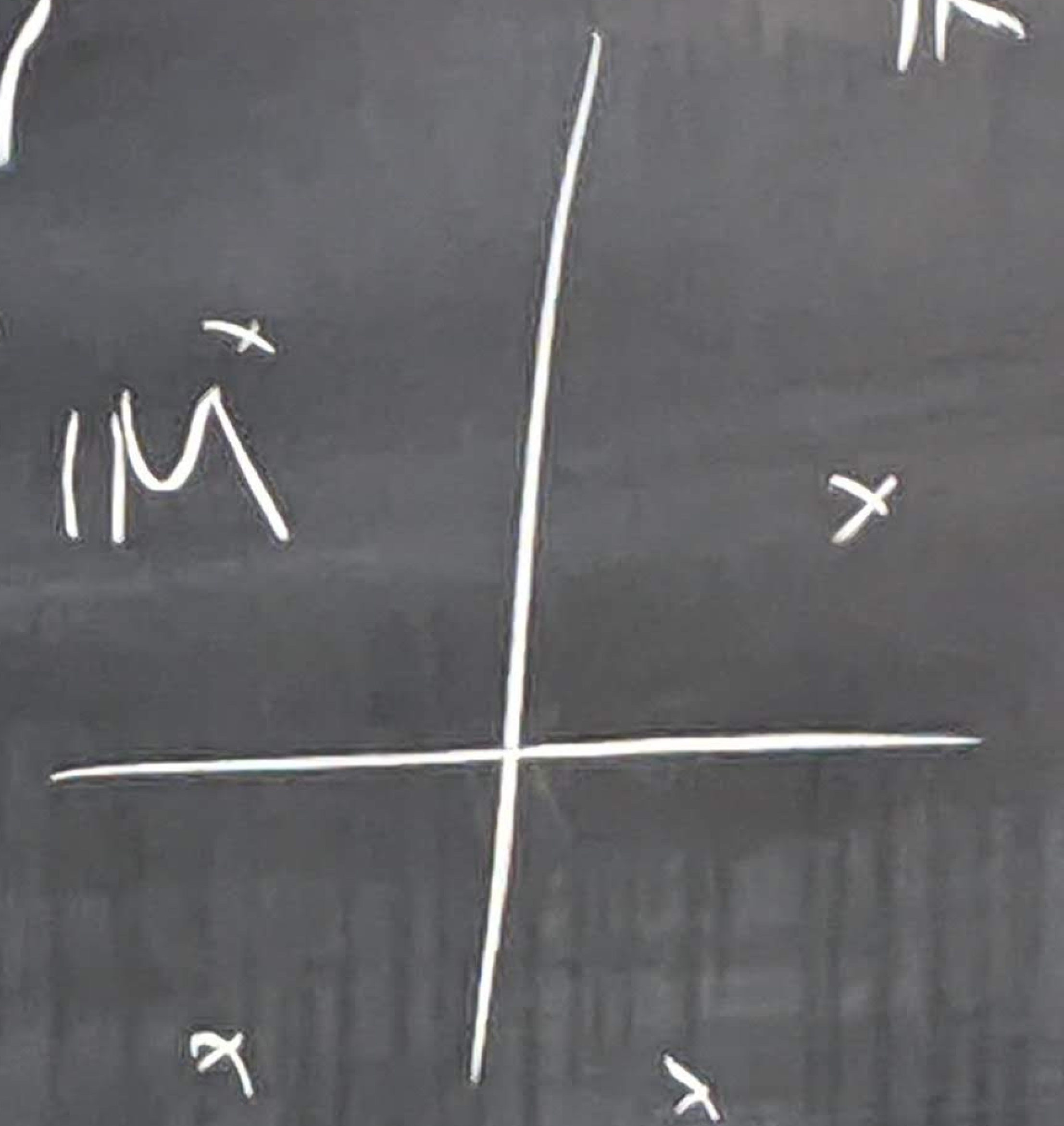
G locally compact second countable group (lcsc)

$\mathbb{M} = \mathbb{M}(G) = \{\omega \subset G \mid \omega \text{ is locally finite}\}$

Π "point process" is a random elt. of $\mathbb{M}$



Π invariant: $g\Pi \overset{(d)}{=} \Pi$

$A \subseteq G$

$N_A(\Pi) = |\Pi \cap A| = \Pi(A)$

$\text{intensity}(\Pi) = \dfrac{\mathbb{E}[N_A(\Pi)]}{\lambda(A)}$, where λ is Haar meas.

Mark

Ξ - c

(Eg.

$\Xi^{\mathbb{M}} =$

For w

Example

Each

indepe

Marked point processes.

Ξ – complete separable metric space of marks/colours

(Eg. $[d]$, $[0,1]$)

$$\Xi^{\mathbb{M}} = \left\{ \omega \in \mathbb{M}(G \times \Xi) \;\middle|\; \begin{array}{l} (g,\xi_1) \in \omega \text{ and } (g,\xi_2) \in \omega \\ \text{then } \xi_1 = \xi_2 \end{array} \right\}$$

For $\omega \in \Xi^{\mathbb{M}}$, write "$g \in \omega$" means $\exists \xi_g \in \Xi$, $(g, \xi_g) \in \omega$.

Example IID marking of Π, denoted $[0,1]^{\Pi}$.

Each $g \in \Pi$ receives a Unif$(0,1)$ label independently. [Bernoulli extension of Π]

Claim

For A

$N_{A \times B}$

For any

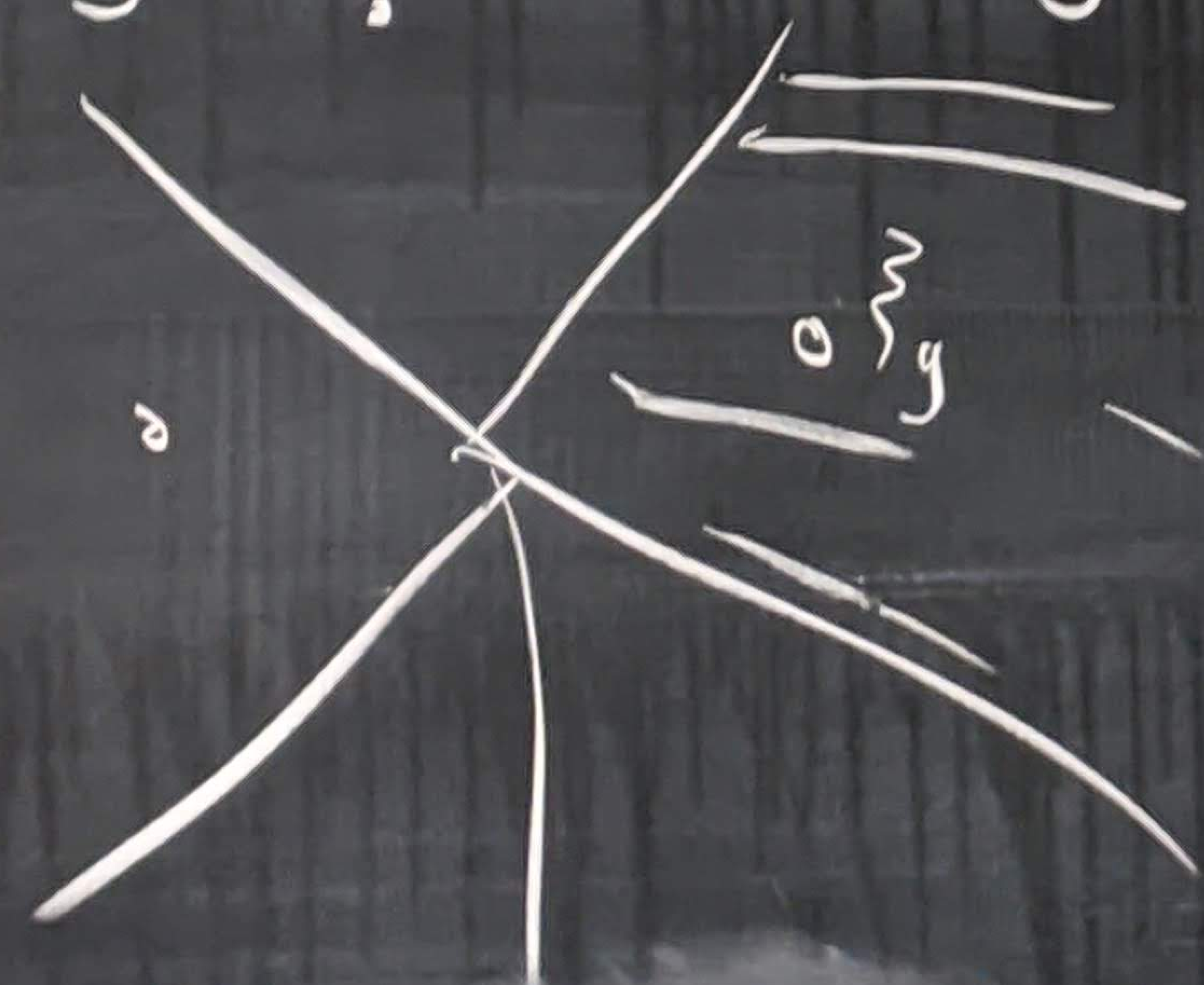
<u>Claim</u> If Ω is invariant so is $[0,1]^{\Omega}$

For $A \subseteq G$, $B \subseteq [0,1]$

$$N_{A\times B}([0,1]^{\Omega}) \quad \text{vs} \quad N_{gA\times B}([0,1]^{\Omega}) \ldots$$

For any invt. Ω, $[0,1]^{\Omega}$ factors onto the Poisson.

Ω



$$([0,1], \mathrm{Leb}) \overset{I}{\cong} (\mathbb{M}, \mathrm{Pois})$$

$$[0,1]^{\Omega} \mapsto \bigcup_{g\in\Omega} \overline{\left(V_{\Omega}(g) \cap I(\xi_g)\right)}$$

If $[0,1]^{\Omega}$ factors onto Υ, then $[0,1]^{\Omega}$ factors on $[0,1]^{\Upsilon}$

$G \curvearrowright (Y, \nu)$ pmp.

Ω p.p. on G w/ law μ

$G \curvearrowright (\mathbb{M} \times Y, \mu \otimes \nu)$ is a Y-marked p.p.

$$\mathbb{M} \times Y \longrightarrow Y^{\mathbb{M}}$$

$$(\omega, y) \mapsto \{(g, g^{-1}y) \in G \times Y \mid g \in \omega\}.$$

$\Gamma \curvearrowright (X, \mu)$ pmp., and $\Gamma < G$ a lattice.

Then $\exists$ pmp action $G \curvearrowright G/\Gamma \times X$ "the induced action"

Take

"lacu

s.t. (

and

$$\Gamma \curvearrowright X \times G \curvearrowleft G$$

$$\Gamma \backslash (X \times G) \curvearrowleft G$$

$$\parallel$$

$$X \times F$$

Take $G \curvearrowright (X,\mu)$ free pmp, it admits a "lacunary"/"cross"-section, that is a Borel $Y \subset X$ s.t. $GY = X$ and $\exists$ compact identity nbhd $U \subset G$ and $Uy \cap Y = \{y\}$ $\forall y \in Y$

p.p.

...ced action"

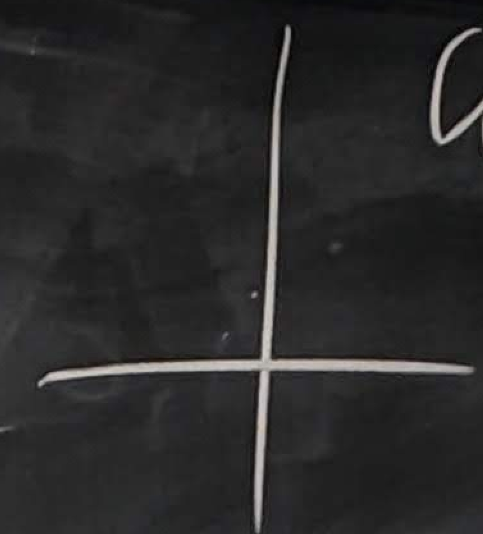
$Gx \cap Y$

Define

and

$\mathbb{R}^2$

X



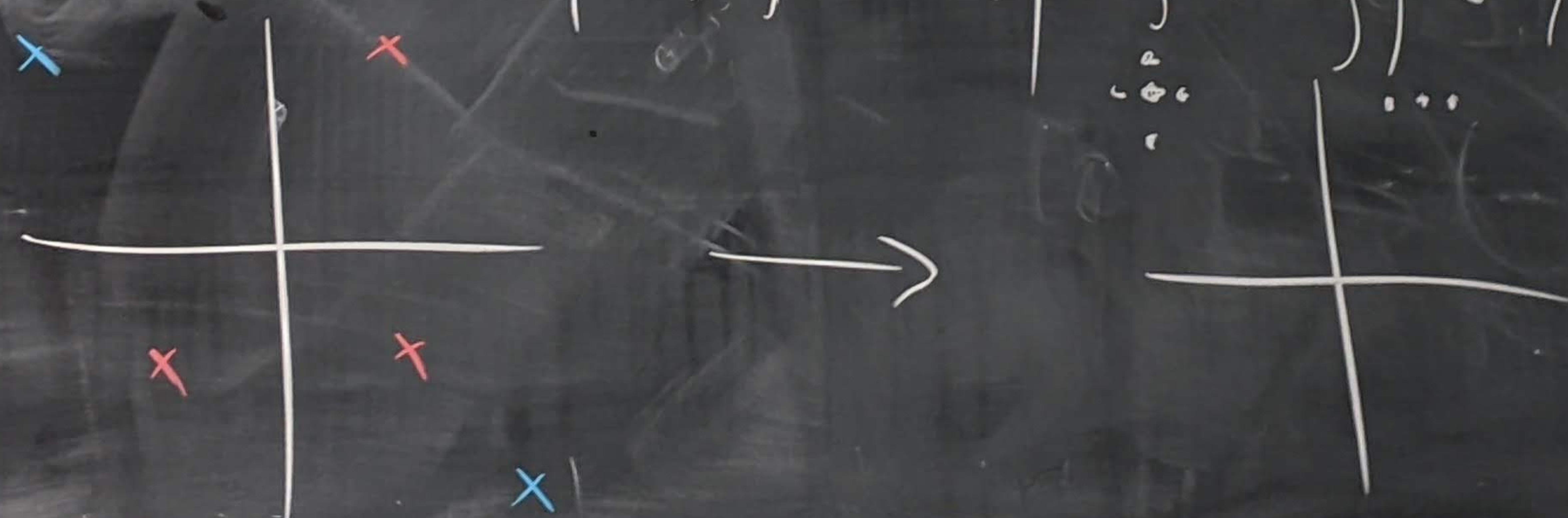
$Gx \cap Y$

Define $V(x) = \{ g \in G \mid g^{-1}x \in Y \} \in \mathbb{M}(G)$

and $\mathcal{V}(x) = \{ (g,y) \in G \times Y \mid g^{-1}x = y \} \in Y^{\mathbb{M}}$

$\mathbb{M}(G)$

$\mathbb{R}^2$



$Y \subset X$

$\in G$